P40: 18: Sol.

$$t\dot{y} + 2y = \sin t \qquad (t>0)$$

$$\Rightarrow \dot{y} + \frac{2}{t}y = \frac{\sin t}{t} \Rightarrow \frac{d}{dt}\left[e^{\int \frac{2}{t}dt} y\right] = e^{\int\frac{2}{t}dt}\frac{\sin t}{t}$$

$$\Rightarrow t^2 y = \int t\sin t\, dt = -t\cos t + \sin t + C$$

$$\Rightarrow y(t) = t^{-2}\left[C - t\cos t + \sin t\right]$$

$$y\left(\frac{\pi}{2}\right) = \frac{4}{\pi^2}\left[C+1\right] = 1 \Rightarrow C = \frac{\pi^2}{4} - 1$$

$$\Rightarrow y(t) = \left[\frac{\pi^2}{4} - 1 - t\cos t + \sin t\right] t^{-2}, \quad (t>0)$$

19. Sol:

$$t^3\dot{y} + 4t^2 y = e^{-t} \qquad (t<0)$$

$$\Rightarrow \dot{y} + \frac{4}{t}y = \frac{e^{-t}}{t^3} \Rightarrow \frac{d}{dt}\left[e^{\int\frac{4}{t}dt} y\right] = t^{-3}e^{-t}e^{\int\frac{4}{t}dt}$$

$$\Rightarrow t^4 y = -(1+t)e^{-t} + C$$

$$y(-1) = 0 \Rightarrow C = 0$$

i.e. $y(t) = -(1+t)t^{-4}e^{-t}$. $(t<0)$.

32. Sol:

$$y(t) = e^{-\frac{t^2}{4}}\int_0^t e^{\frac{s^2}{4}}ds + Ce^{-\frac{t^2}{4}} =: \mathrm{I} + \mathrm{II}$$

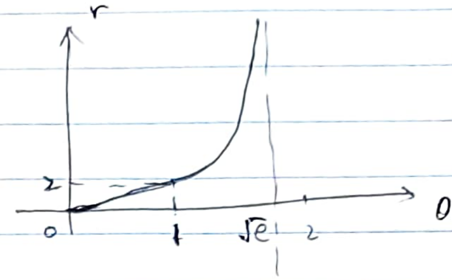
$\mathrm{II} \to 0$ as $t \to \infty$

$$\mathrm{I} = \frac{\int_0^t e^{\frac{s^2}{4}}ds}{e^{\frac{t^2}{4}}} \qquad \lim_{t\to\infty}\mathrm{I} = \lim_{t\to\infty}\frac{e^{\frac{t^2}{4}}}{e^{\frac{t^2}{4}}\cdot 2t} = \lim_{t\to\infty}\frac{1}{2t} = 0.$$

i.e. $\lim_{t\to\infty} y(t) = 0$.

Pa8. 12. Sol:

a> $\frac{dr}{d\theta} = \frac{r^2}{\theta} \Rightarrow \frac{dr}{r^2} = \frac{d\theta}{\theta}$



$\Rightarrow -\frac{1}{r} = \log|\theta| + c \Rightarrow r = \frac{1}{c - \log|\theta|}$

$r(1) = 2 \Rightarrow c = \frac{1}{2} \Rightarrow r(\theta) = \frac{2}{1 - 2\log|\theta|}$

c> $r(\theta) \neq 0 \Rightarrow |\theta| \neq \sqrt{e} > 1$

$r(\theta)$ is well defined $\Rightarrow 0 < \theta < \sqrt{e}$.

29. Sol:

$$\frac{dx}{dy} = \frac{cy + d}{ay + b} = \frac{c}{a} + \frac{d - \frac{bc}{a}}{ay + b}$$

$\Rightarrow x(y) = \frac{c}{a} y + \frac{ad - bc}{a^2} \log|ay + b| + k.$ $(a \neq 0,\ ay + b \neq 0,\ k \text{ const})$

30. Sol:

a> $\frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$ $(x \neq 0)$

b> $v := \frac{y}{x} \Rightarrow \frac{dv}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}\left(\frac{dy}{dx} - v\right)$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{v - 4}{1 - v}$

d> $x\frac{dv}{dx} = \frac{v^2 - 4}{1 - v} \Rightarrow \frac{dv}{\left(\frac{v^2 - 4}{1 - v}\right)} = \frac{dx}{x}$

$\Rightarrow \int \frac{1 - v}{v^2 - 4}\, dv = \log|x| + c$

$\Rightarrow -\frac{1}{4}\left\{\log|v - 2| + 3\log|v + 2|\right\} = c + \log|x|$

$\Rightarrow \log\left[|v - 2||v + 2|^3\right] = c - \log|x|^4$

e> $v = \frac{y}{x} \Rightarrow \log\left[|y - 2x||y + 2x|^3\right] = c$

$\Rightarrow |y - 2x||y + 2x|^3 = c$

P76 5. Sol:

$-2 < t < 2$

24. Sol:

$$\frac{d}{dt}\left[c\varphi(t)\right] + p(t)\left[c\varphi(t)\right]$$

$$= c\left[\dot{\varphi}(t) + p(t)\varphi(t)\right] = 0.$$

25. Sol:

$$\frac{d}{dt}\left[y_1(t) + y_2(t)\right] + p(t)\left[y_1(t) + y_2(t)\right] = g(t).$$

26. Sol:

a> $y(t) = \frac{c}{\mu(t)} + \frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)g(s)\,ds.$

b> $\frac{d}{dt}\left(\frac{1}{\mu(t)}\right) + p(t)\cdot\frac{1}{\mu(t)}$

$$= \frac{d}{dt}\left\{e^{-\int_{t_0}^{t}p(s)ds}\right\} + p(t)\,e^{-\int_{t_0}^{t}p(s)\,ds}$$

$$= e^{-\int_{t_0}^{t}p(s)ds}\left(-p(t) + p(t)\right) = 0.$$

c> $\frac{d}{dt}\left\{\frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)g(s)\,ds\right\} + p(t)\cdot\frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)g(s)\,ds$

$$= e^{-\int_{t_0}^{t}p(s)ds}\left\{-p(t)\int_{t_0}^{t}\mu(s)g(s)ds + \mu(t)g(t) + p(t)\int_{t_0}^{t}\mu g\,ds\right\}$$

$$= g(t)$$

29. Sol:

set $v := y^{-1}$, then $\frac{dy}{dt} = \frac{d(v^{-1})}{dt} = -\frac{1}{v^2}\frac{dv}{dt}$

$$\Rightarrow -\frac{1}{v^2}\frac{dv}{dt} = rv^{-1} - kv^{-2} \Rightarrow \frac{dv}{dt} + rv - k = 0$$

i.e. $v(t) = e^{-rt}\left(\frac{k}{r}e^{rt} + c\right) = \frac{1}{r}\left\{cre^{-rt} + k\right\}$

$$\Rightarrow y(t) = \frac{r}{k + cre^{-rt}}.$$

P121. 15. Sol:

$$f(t,y_1)-f(t,y_2) = \int_{y_1}^{y_2} \frac{d}{ds} f(t,s)\,ds$$

$$= \int_{y_1}^{y_2} \partial_y f(t,s)\,ds \le \sup_{(t,s)\in D} |\partial_y f(t,s)| \cdot |y_2-y_1|$$

$\partial_y f \in C(D) \Rightarrow \partial_y f$ is bdd in D

set $K := \sup_{(t,s)\in D} |\partial_y f(t,s)| \Rightarrow |f(t,y_1)-f(t,y_2)| \le K|y_1-y_2|$.

16. Sol:

Only need to show $(t,\varphi_n(t)) \in D$, $\forall n$,

indeed, take $|h| \le \frac{b}{\max_R |f|}$, then

$$|\varphi_{n+1}(t)| \le \int_0^t |f(s,\varphi_n(s))|\,ds \le t \max_R |f| \le b, \text{ if } |t|\le h.$$

then $|f(t,\varphi_n(t)) - f(t,\varphi_{n-1}(t))| \le K|\varphi_n(t)-\varphi_{n-1}(t)|$.

17. Sol:

a> trivial

b> $|f(t,\varphi_1(t)) - f(t,\varphi_0(t))| \le K|\varphi_1(t)-\varphi_0(t)| \le KM|t|$

$$\Rightarrow |\varphi_2(t)-\varphi_1(t)| \le \int_0^t |f(s,\varphi_1(s))-f(s,\varphi_0(s))|\,ds \le \frac{KM|t|^2}{2}$$

c> Suppose it holds for $n\le k$, then

$$|\varphi_{k+1}(t)-\varphi_k(t)| \le \int_0^t |f(s,\varphi_k(s))-f(s,\varphi_{k-1}(s))|\,ds$$

$$\le \int_0^t K\frac{MK^{k-1}|s|^k}{k!}\,ds \le \frac{MK^k|t|^{k+1}}{(k+1)!} \le \frac{MK^k h^{k+1}}{(k+1)!}$$

i.e. it holds for all $n\in\mathbb{N}$.

18. Sol:

a> trivial

b> trivial

c> $|\varphi_n(t)| \le \sum_{m=0}^{\infty} \frac{M}{K}\left(\frac{(Kh)^m}{m!}\right) = \frac{M}{K}e^{Kh} < \infty$.

19. Sol: Omitted.